**Phys 410**
**Fall 2015**
**Lecture #12 Summary**
**8 October, 2015**

Going back to the shortest-distance-in-a-plane problem, we see that the function $f$ in this case is $f = \sqrt{1+(y')^2}$. In this case $f$ does not depend explicitly on$y$, hence we can write $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}} = C$, a constant. This can be reduced to $y'(x) = m$, where $m$ is another constant. Integrating both sides with respect to $x$, we find $y(x) = mx + b$, which is the famous equation for a straight line. Hence the shortest distance between two points in a flat plane is a straight line. The Fermat's principle problem can be solved in a similar way once the index of refraction distribution $n(x, y)$, and the end points, are specified.

We then did the example of the Brachistochrone problem. A particle falls from rest under the influence of gravity following a frictionless track to a final location. The question is: what track design will get the particle to the final location in the shortest time? The particle starts at the origin (x=0, y=0) and falls to a point ($x_2$, $y_2$), with $x_2 > 0$ and $y_2 > 0$ (note that positive y is in the 'down' direction). The time to travel is given by $Time(1 \to 2) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{dx^2+dy^2}}{v}$. The speed is found from conservation of energy: $v = \sqrt{2gy}$, leading to $Time(1 \to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{1+(x')^2}}{\sqrt{y}} dy$, where we are using the y-coordinate of the particle as the independent variable and $x' = dx/dy$. We are now looking for the trajectory $x(y)$ that minimizes the time: $Time(1 \to 2)$. This integral will be made stationary when the integrand $f(x, x', y)$ obeys the Euler-Lagrange equation, which in this case is: $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$. The result is a differential equation for $x(y)$: $x' = \sqrt{\frac{y}{2a-y}}$, where $a$ is a constant introduced from the Euler-Lagrange equation. We can integrate this equation with the change of variables $y = a(1-\cos\theta)$, yielding $x = a(\theta - \sin\theta) + C$. This describes a cycloid curve (our cycloid is an upside-down version of the one on this web site). The particle making the shortest-time fall will follow the cycloid trajectory.

In general, it is not always possible to parameterize the trajectory of a particle with a simple one-to-one functional relationship such as $y(x)$ or $x(y)$. In this case one would like to parameterize the trajectory with functions such as $(x(u), y(u))$, where $u$ acts as the parameter. The Euler-Lagrange equation can be generalized to handle this situation. Consider the integral $S = \int_{u_1}^{u_2} f[x(u), x'(u), y(u), y'(u), u]du$. To make it stationary will yield two Euler-Lagrange equations: $\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0$ and $\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0$.

We then showed that Newton's second law of motion can be re-stated as a set of Euler-Lagrange equations for an integrand known as the Lagrangian $\mathcal{L} = T - U$, where $T$ is the kinetic energy and $U$ is the potential energy. The integral that is made stationary is called the action: $S = \int \mathcal{L}\, dt$. Hamilton's principle states that the actual motion of the particle will be the one that leaves this integral stationary. The Lagrangian can be written in terms of any set of unique (generalized) coordinates $(q_1, q_2, q_3)$. One can define a generalized force as $\frac{\partial \mathcal{L}}{\partial q_i}$, and the generalized momentum as $\frac{\partial \mathcal{L}}{\partial q_i'}$. They are related through the Euler-Lagrange equation as "generalized force" = time rate of change of "generalized momentum". Note that these generalized quantities do not necessarily have the dimensions of force or momentum!

Feynman's path integral formulation of quantum mechanics considers all possible trajectories between the initial point and the final point. One calculates a transition amplitude as a sum over all trajectories of a weighting function. The weight of each trajectory is given the same magnitude, but a variable phase, as $e^{iS/\hbar}$, where $S$ is the action for that trajectory and $\hbar$ is Planck's constant divided by $2\pi$, which is sometimes known as the quantum of action. This is a generalization of Hamilton's principle, which of course specifies only a single classical trajectory.

We considered the Lagrangian in polar coordinates for a single particle of mass $m$ acted upon by a conservative force in two dimensions. The Lagrangian is $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$. The Euler-Lagrange equation for $r$ yields $-\frac{\partial U}{\partial r} = m(\ddot{r} - r\dot{\phi}^2)$. This is Newton's second law for radial motion, where the first term on the right hand side is the radial acceleration, while the second is the centripetal acceleration. The Euler-Lagrange equation for $\varphi$ yields $-\frac{\partial U}{\partial \varphi} = \frac{d}{dt}(mr^2\dot{\varphi})$. This is a statement that the torque acting on the particle $\left(-\frac{\partial U}{\partial \varphi} = rF_\varphi\right)$ is equal to the time rate of change of the angular momentum. In other words it is a statement of Newton's second law for rotational motion.